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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO MULTIVALUED DIFFERENTIAL SYSTEMS IN INFINITE DIMENSIONAL SPACES

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INTRODUCTION

1. HISTORY AND SIGNIFICANCE OF THE PROBLEM

Evolution inclusions emerge from various problems, including control problems with multivalued feedbacks, differential equations with discontinuous right-hand side, and differential variational inequalities. The study of asymptotic behavior of solutions to evolution inclusions in this thesis consists of the (weak) stability for stationary points, the existence of global attractor for the associated dynamical system, and some classes of special solutions such as anti-periodic and decay solutions.

Evolution inclusions in finite dimensional spaces have been studied early. The solvability and structure of solution set were presented systematically in the monograph of Deimling (1992). Subsequently, evolution inclusions in Banach spaces and their applications became an important subject for researchers in the last decades. We refer the reader to monographs of Tolstonogov (2000) and Kamenskii et al. (2001).

One of the most important questions in the study of differential equations is the stability of solutions. For ordinary differential equations, the classical Lyapunov theory has been an effective tool to address the stability of their solutions. In order to attack the stability of solutions to partial differential equations, the theory of global attractors was introduced.

The Lyapunov theory and the framework for studying global attractors have been developed to deal with the stability of solutions to evolution inclusions. Since the uniqueness for Cauchy problem associated with evolution inclusions is unavailable, the classical Lyapunov theory does not work for studying the stability of solutions. As far as the evolution inclusions in finite dimensional spaces are concerned, the concept of weak stability was introduced by Filippov (1988). Regarding the evolution inclusions

in infinite dimensional spaces, the most frequently used technique was attractor theory.

In recent decades, attractor theory has been well-developed and systematic results have been achieved (see the monographs of Raugel (2002) and Babin (2006). Regarding the behavior of multivalued dynamical systems associated to differential equations without uniqueness or differential inclusions, some famous theories such as the theory of $m-$semiflows established by Melnik and Valero (1998), and the theory of generalized semiflows given by Ball (1997) have been used. A comparison of these two theories was given by Carabalo (2003). In the sequel, the concepts of pullback attractor and uniform attractor were also introduced to deal with non-autonomous evolution inclusions (see Carabalo et al. (1998; 2003), Melnik and Valero (2000)). Especially, in the last two years, some remarkable improvements for the theory of global attractors were made by Kalita et al.. The latest results on global attractors focus on relaxing the continuity conditions and giving criteria for asymptotic compactness of semigroups/processes based on the measure of noncompactness. However, applying these criteria to functional differential systems is difficult due to the complication of associated phase spaces.

Thanks to the framework of Melnik and Valero, in this thesis, we study the existence of a compact global attractor for the m-semiflow generated by the problem

$$u'(t) \in Au(t) + F(u(t), u_t), \quad t \geq 0, \tag{1}$$

$$u(s) = \varphi(s), \quad s \in [-h, 0], \tag{2}$$

where u is the state function with values in X , u_t stands for the history of the state function up to time t, i.e. $u_t(s) = u(t+s)$ for $s \in [-h, 0]$, F is a multivalued map defined on a subset of $X \times C([-h, 0]; X)$. In this model, $A : D(A) \subset X \to X$ is a linear operator satisfying the Hille–Yosida condition but $\overline{D(A)} \neq X$.

For the fractional differential/inclusions equations, since the semigroup property does not hold in their solution set, the theory of global attractors is useless in studying the asymptotic behavior of solutions. Moreover, the classical concept of Lyapunov stability theory cannot be applied to multi-valued cases. Therefore, we adopt the concept of *weakly asymptotic stability of zero solution* when studying the class of fractional inclusions:

$$D_0^\alpha u(t) \in Au(t) + F(t, u(t), u_t),\ t > 0, t \neq t_k, k \in \Lambda, \tag{3}$$

$$\Delta u(t_k) = I_k(u(t_k)), \tag{4}$$

$$u(s) + g(u)(s) = \varphi(s),\ s \in [-h, 0], \tag{5}$$

where $D_0^\alpha, \alpha \in (0,1)$, is the fractional derivative in the Caputo sense, A is a closed linear operator in X which generates a strong continuous semigroup $W(\cdot)$, $F : \mathbb{R}^+ \times X \times C([-h,0]; X) \to \mathcal{P}(X)$ is a multivalued map, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $k \in \Lambda \subset \mathbb{N}$, I_k and g are the continuous functions. Here u_t stands for the history of the state function up to the time t.

The system (3)-(5) is a generalized Cauchy problem which involves impulsive effect and nonlocal condition expressed by (4) and (5), respectively. In the case $\alpha = 1$, the problem with nonlocal and impulsive conditions has been studied extensively. It is known that nonlocal conditions give a better description for real models than classical initial ones, e.g., the condition

$$u(s) + \sum_{i=1}^{M} c_i u(\tau_i, s) = \varphi(s)$$

allows taking some measurements in addition to solely initial one. On the other hand, impulsive conditions have been used to describe the dynamical systems with abrupt changes. There have been extensive studies devoted to particular cases of this problem in literature. We refer to some typical results on the existence

and properties of solution set presented by A. Cernea (2012), R.N. Wang et al. (2014, 2015), M. Feckan et al. (2015), in which the solvability on compact intervals and the structure of solution set like R_δ-set were proved. Regarding related control problems, it should be mentioned the results on controllability given by J.R.Wang and Y. Zhou (2011), R. Sakthivel, R. Ganesh and S.M. Anthoni (2013), R.N.Wang, Q.M. Xiang and P.X. Zhu (2014). One of the most important questions in the problem (3)-(5) is to analyze the stability of its solutions. Unfortunately, the results on this direction are less known.

Together with stability theory, finding special classes such as anti-periodic solutions of differential system also attracts many researchers. The existence of anti-periodic solutions to nonlinear evolution equations has been investigated by many authors in the last decades since the work of H. Okochi (1988) (see also H. Okochi (1990)). Without stressing to widen the list of references, we quote here some remarkable results of A. Haraux (1989), Y. Wang (2010), Z.H. Liu (2010). Recently, Q. Liu (2012) has dealt with the existence of the anti-periodic mild solutions to the semilinear abstract differential equation in the form

$$\begin{aligned} u'(t) + Au(t) &= f(t, u(t)), t \in \mathbb{R}, \\ u(t+T) &= -u(t), t \in \mathbb{R}, \end{aligned}$$

where $\mathbb{R}$ stands for the set of real numbers and A is the generator of a hyperbolic C_0-semigroup. Since this work, the existence of anti-periodic solutions to differential equations in Banach spaces by using semigroup theory has been established by many authors, for example, we refer readers to the results of D. O'Regan et al. (2012), R. N. Wang and D. H. Chen (2013), V. Valmorin (2012), J.H. Liu et al. (2014, 2015).

All the results about the solution of anti-periodic problem are, however, in the equation form and most of them need Lipschitz condition for the nonlinear part in right hand side. Therefore, in

this thesis, we study the existence of anti-periodic solutions to a class of polytope differential inclusions

$$u'(t) \in Au(t) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{6}$$

$$u(t+T) = -u(t), t \in \mathbb{R}, \tag{7}$$

where $F(t, u(t)) = \overline{\text{conv}}\{f_1(t, u(t)), \cdots, f_n(t, u(t))\}$; A is a Hille-Yosida operator having the domain $D(A)$ such that $\overline{D(A)} \neq X$ and the part of A in $\overline{D(A)}$ generates a hyperbolic semigroup.

Because of these, we select the above subjects for the main content of the thesis: "Asymptotic behavior of solution to evolution inclusions in infinite dimensional space".

2. PURPOSES, OBJECTS AND SCOPE OF THE THESIS

The thesis focuses on studying the solvability and asymptotic behavior of some classes of differential inclusions in infinite dimensional spaces. More precisely as follows.

- **Content 1:** The existence of global attractors for multivalued dynamics generated by semilinear functional evolution inclusions.
- **Content 2:** The existence of anti-periodic solutions to semilinear evolution inclusions.
- **Content 3:** The weak stability of stationary solutions to semilinear evolution inclusions.

3. METHOD OF THE THESIS

- To study the solvability, we employ the semigroup method, MNC estimate method and fixed points theory.
- To prove the existence of global attractors for multivalued dynamics generated by semilinear functional evolution in-

clusions, we employ the frameworks of Melnik and Valero (1998).

- To analyze the weak stability of stationary solutions to semilinear evolution inclusions, we make use of the fixed point techniques.

4. RESULTS AND TRUCTURE OF THESIS

Together with the Introdution, Inclusion, Author's works related to the thesis that have been published and References, the thesis includes four chapters:

- Chapter 1: *Preliminaries.* This chapter presents the basic notions and known results of the general theory of semigroup, measure of noncompactness, condensing maps, fractional calculus and global attractor of $m-$semiflows.

- Chapter 2: *Global attractor for a class of functional differential inclusions.* This chapter devotes to prove the global solvability and the existence of a compact global attractor for the m-semiflow generated by a class of functional differential inclusions with Hille–Yosida operators.

- Chapter 3: *Existence of anti-periodic solutions for a class of polytope differential inclusions.* In this chapter, we prove the existence of anti-periodic solutions for a class of polytope differential inclusions assuming that its linear part is a non-densely defined Hille-Yosida operator.

- Chapter 4: *Weak stability for a class of semilinear fractional differential inclusions.* In this chapter, we prove the global solvability and weak asymptotic stability for a semilinear fractional differential inclusion subject to impulsive effects and nonlocal condition.

Chapter 1

PRELIMINARIES

This chapter presents some preliminaries including: some functional spaces; semigroup theory; measure of noncompactness; fixed points theorem for multivalued maps; global attractor of $m-$semiflows and fractional calculus.

1.1. Some functional spaces

In this section, we recall some functional spaces and functional spaces depending on time which will be used in our thesis.

1.2. Semigroup

In this section, we present the basic knowledge about semigroup theory and some common semigroup, especially the insight into integrated semigroup.

1.3. Measure of noncompactness (MNC) and MNC estimate

In this section, we recall some notions and facts related to measure of noncompactness (MNC) and Hausdorff MNC, followed by some MNC estimate which is necessary for the next chapters.

1.4. Condensing map and fixed points theorem for multivalued maps

In this section, we recall some notions of set-valued analysis and condensing map, then introduce some fixed point theorem for multivalued maps.

1.5. Global attractor of $m-$semiflows

In this section, we present theory of global attractor of $m-$semiflows of Melnik and Valero (1998) and the framework to prove the existence of a compact global attractor for $m-$semiflows generated by a differential inclusions.

1.6. Fractional calculus

In this section, we recall some notions and facts related to fractional calculus, fractional resolvent operators.

Chapter 2

GLOBAL ATTRACTOR FOR A CLASS OF FUNCTIONAL DIFFERENTIAL INCLUSIONS

We study the dynamics for a class of functional differential inclusions whose linear part generates an integrated semigroup. Some techniques of measure of noncompactness are deployed to prove the global solvability and the existence of a compact global attractor for the m-semiflow generated by our system. The obtained results generalize recent ones in the same direction.

The content of this chapter is written based on the paper [2] in the author's works related to the thesis that has been published.

2.1. Setting problem

Let $(X, \|\cdot\|)$ be a Banach space, we consider the following problem

$$u'(t) \in Au(t) + F(u(t), u_t), \quad t \geq 0, \tag{2.1}$$

$$u(s) = \varphi(s), \quad s \in [-h, 0], \tag{2.2}$$

where u_t stands for the history of the state function up to time t, i.e. $u_t(s) = u(t+s)$ for $s \in [-h, 0]$, F is a multivalued map defined on a subset of $X \times C([-h, 0], X)$. In this model, $A : D(A) \subset X \to X$ is a linear operator satisfying the Hille-Yosida condition.

2.2. Existence of integral solution

Denote

$$\mathcal{P}_c(X) = \{D \in \mathcal{P}(X) : D \text{ is closed}\},$$

$$\mathcal{C}_h = \{\varphi \in C([-h, 0]; X) : \varphi(0) \in \overline{D(A)}\},$$

$$\mathcal{C}_\varphi = \{v : J \to \overline{D(A)}, v \in C(J, X), v(0) = \varphi(0)\}.$$

For $v \in \mathcal{C}_\varphi$, we denote $v[\varphi] \in C([-h,T],X)$ as follows

$$v[\varphi](t) = \begin{cases} v(t) \text{ if } t \in [0,T] \\ \varphi(t) \text{ if } t \in [-h,0]. \end{cases}$$

In what follows, we use the assumption that

(**A**) *The operator A satisfies the Hille-Yosida condition and the C_0-semigroup $\{S'(t)\}_{t\geq 0}$ is norm-continuous.*

(**F**) *The multi-valued function $F : \overline{D(A)} \times \mathcal{C}_h \to \mathcal{P}_c(X)$ satisfies:*

(1) *F is u.s.c with weakly compact and convex values;*

(2) *$\|F(x,y)\| := \sup\{\|\xi\| : \xi \in F(x,y)\} \leq a\|x\| + b\|y\|_{\mathcal{C}_h} + c$, for all $x \in \overline{D(A)}$, $y \in \mathcal{C}_h$, where $a,b,c \geq 0$;*

(3) *if $\{S'(t)\}$ is noncompact, then $\chi(F(B,C)) \leq p\chi(B) + q \sup\limits_{t\in[-h,0]} \chi(C(t))$, for all $B \subset \overline{D(A)}, C \subset \mathcal{C}_h$, where $p,q \in \mathbb{R}^+$.*

Putting $\mathcal{P}_F(v) = \{f \in L^1(J;X) : f(t) \in F(v(t), v[\varphi]_t),$ a.e. $t \in J\}$, we have the definition of integral solution to our problem.

Definition 2.1. For a given $\varphi \in \mathcal{C}_h$, a continuous function $u : [-h,T] \to X$ is said to be an integral solution to problem (2.1)-(2.2) on $[-h,T]$ with initial datum φ if $\exists f \in \mathcal{P}_F(u)$ such that

$$u(t) = \begin{cases} S'(t)\varphi(0) + \lim\limits_{\lambda\to+\infty} \int_0^t S'(t-s)\lambda(\lambda I - A)^{-1} f(s)ds, t \geq 0, \\ \varphi(t), t \in [-h,0]. \end{cases}$$

Theorem 2.1. *Let the hypotheses* (**A**) *and* (**F**) *hold. Then problem* (2.1)-(2.2) *has at least one integral solution for each initial datum $\varphi \in \mathcal{C}_h$.*

2.3. Existence of global attractor

The m-semiflow governed by (2.1)-(2.2) is defined as follows

$G : \mathbb{R}^+ \times \mathcal{C}_h \to \mathcal{P}(\mathcal{C}_h)$,
$G(t, \varphi) = \{u_t : u[\varphi] \text{ is an integral solution of } (2.1) - (2.2)\}$.

In this section, we need an additional assumption as following.

(S) $\exists \alpha, \beta > 0, N \geq 1$ such that

$$\|S'(t)\|_{\mathcal{L}(X)} \leq e^{-\alpha t}, \ \|S'(t)\|_{\chi} \leq N e^{-\beta t}, \forall t > 0.$$

Theorem 2.2. *Let the hypotheses* (**A**), (**F**) *and* (**S**) *hold. Then the m-semiflow G generated by system* (2.1)-(2.2) *admits a compact global attractor provided that*

$$\min\{\alpha - (a + b), \beta - 4N(p + q)\} > 0.$$

2.4. Application

2.4.1. Partial differential inclusion in bounded domain

Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $\mathcal{O} \subset \Omega$ be an open subset. Consider the following problem (I)

$$\frac{\partial u}{\partial t}(t, x) - \Delta_x u(t, x) + \lambda u(t, x) = f(x, u(t, x)) + \sum_{i=1}^{m} b_i(x) v_i(t), x \in \Omega, t > 0,$$

$$v_i(t) \in \left[\int_{\mathcal{O}} k_{1,i}(y) u(t - h, y) dy, \int_{\mathcal{O}} k_{2,i}(y) u(t - h, y) dy\right], 1 \leq i \leq m,$$

$$u(t, x) = 0, \ x \in \partial\Omega, t \geq 0,$$

$$u(s, x) = \varphi(s, x), \ x \in \Omega, s \in [-h, 0],$$

where $\lambda > 0$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfies $|f(x, r)| \leq a(x)|r| + b(x), \forall x \in \Omega, r \in \mathbb{R}$, $b_i \in C(\overline{\Omega}), k_{j,i} \in L^1(\mathcal{O})$ for $i \in \{1, ..., m\}, j = 1, 2$, and $\varphi \in \mathcal{C}_h = C([-h, 0]; C(\overline{\Omega}))$. Let

$$X = C(\overline{\Omega}), X_0 = C_0(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\},$$

are endowed with the sup norm $\|v\| = \sup_{x\in\Omega} |v(x)|$. So following Theorem 2.2, the m-semiflow generated by (I) has a compact global attractor in $C([-h,0]; C(\overline{\Omega}))$ if

$$\|a\| + \sum_{i=1}^{m} \|b_i\| \max\{\int_{\mathcal{O}} |k_{1,i}(y)|dy; \int_{\mathcal{O}} |k_{2,i}(y)|dy\} < \lambda.$$

2.4.2. Partial differential inclusion in unbounded domain

We consider the following problem (II) with $\Omega = \mathbb{R}^n$ and $\mathcal{O}$ is a bounded domain in $\mathbb{R}^n$

$$\frac{\partial u}{\partial t}(t,x) - \Delta_x u(t,x) + \lambda u(t,x) = f(x, u(t,x)) + \sum_{i=1}^{m} b_i(x) v_i(t), x \in \mathbb{R}^n, t > 0,$$

$$v_i(t) \in \left[\int_{\mathcal{O}} k_{1,i}(y) u(t-h,y)dy, \int_{\mathcal{O}} k_{2,i}(y) u(t-h,y)dy\right], 1 \le i \le m,$$

$$u(s,x) = \varphi(s,x),\ x \in \mathbb{R}^n, s \in [-h,0].$$

In this model, we assume that

1) $b_i \in L^2(\mathbb{R}^n)$, $k_{j,i} \in L^2(\mathcal{O}), j = 1,2; 1 \le i \le m$ and $\varphi \in C([-h,0]; L^2(\mathbb{R}^n))$;

2) $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $f(\cdot, z)$ is measurable for each $z \in \mathbb{R}$ and there exists $\kappa \in L^2(\mathbb{R}^n)$ verifying

$$|f(x,z_1) - f(x,z_2)| \le \kappa(x)|z_1 - z_2|, \forall x \in \mathbb{R}^n, z_1, z_2 \in \mathbb{R}.$$

Let $X = L^2(\Omega)$, we have $A = \Delta - \lambda I$ generates a analytic semigroup $T(\cdot)$, which $T(\cdot)$ is exponentially stable and χ-decreasing with exponent λ. We have the following result due to Theorem 2.2.

Theorem 2.3. *The m-semiflow generated by (II) admits a compact global attractor in $C([-h,0]; L^2(\mathbb{R}^n))$ provided that*

$$\max\{4\|\kappa\|, \|\kappa\| + \sum_{i=1}^{m} \|b_i\| \max\{\|k_{1,i}\|_{L^2(\mathcal{O})}, \|k_{2,i}\|_{L^2(\mathcal{O})}\} < \lambda.$$

Chapter 3

EXISTENCE OF ANTI-PERIODIC SOLUTIONS FOR A CLASS OF POLYTOPE DIFFERENTIAL INCLUSIONS

In this section, we prove the existence of anti-periodic solutions for a class of polytope differential inclusions in Banach space assuming that its linear part is a non-densely defined Hille-Yosida operator.

The content of this chapter is written based on the paper [1] in the author's works related to the thesis that has been published.

3.1. Setting problem

Let $(X, \|\cdot\|)$ be a Banach space. In this chapter, we are concerned with the existence of the solution for the following problem

$$u'(t) \in Au(t) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{3.1}$$

$$u(t+T) = -u(t), t \in \mathbb{R} \tag{3.2}$$

where $F(t, u(t)) = \overline{\text{conv}}\{f_1(t, u(t)), \ldots, f_n(t, u(t))\}$; A is a Hille-Yosida operator having the domain $D(A)$ such that $\overline{D(A)} \neq X$ and the part of A in $\overline{D(A)}$ generates a hyperbolic semigroup.

3.2. Existence of anti-periodic mild solution

Denote $P_{TA}(\mathbb{R}; X) = \{u \in BC(\mathbb{R}; X) : u(t+T) = -u(t)\}$, it is easy to see that $P_{TA}(\mathbb{R}; X)$, equipped with the sup normed, is a Banach space. We assume that:

(**A**) *The operator A satisfies the Hille-Yosida condition. In addition, $\{S'(t)\}_{t\geq 0}$ is hyperbolic.*

(**F**) *The function $f_i : \mathbb{R} \times \overline{D(A)} \to X, i = 1, \cdots, n$ satisfies:*

(1) $f_i(\cdot, x)$ *is strongly measurable for every* $x \in \overline{D(A)}$ *and* $f_i(t, \cdot)$ *is continuous for a.e.* $t \in \mathbb{R}$*;*

(2) $\|f_i(t,x)\| \leq m(t)(\|x\| + 1)$*, for all* $x \in \overline{D(A)}$*, where* $m \in L^1_{loc}(\mathbb{R}; \mathbb{R}^+)$*;*

(3) if $S'(\cdot)$ *is noncompact, then* $\chi(f_i(t, B)) \leq k(t)\chi(B)$*, for all* $B \subset \overline{D(A)}$*, where* $k \in L^1_{loc}(\mathbb{R}; \mathbb{R}^+)$*,*

(4) $f_i(t + T, -x) = -f_i(t, x)$ *for all* $x \in \overline{D(A)}$*.*

Definition 3.1. A mild solution to equation (2.1)-(2.2) is a function $u \in BC(\mathbb{R}; X)$ satisfying the integral equation

$$u(t) = S'(t-s)u(s) + \lim_{\lambda \to +\infty} \int_s^t S'(t-s) R_\lambda f(s) ds,$$

where $R_\lambda = \lambda(\lambda I - A)^{-1}$, for all $t > s$ and $s \in \mathbb{R}, f \in \mathcal{P}_F^{TA}(u)$.

Theorem 3.1. *Let the hypotheses* (**A**) *and* (**F**) *hold. Then problem* (2.1)-(2.2) *has at least one integral solution provided that*

$$\frac{2N}{1 - e^{-\delta T}} \int_0^T m(s) ds < 1. \tag{3.3}$$

3.3. Applications

3.3.1. Example 1

Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the following problem

$$\frac{\partial u}{\partial t}(t,x) - \Delta_x u(t,x) + \lambda u(t,x) = f(t, x, u(t,x)), x \in \Omega, t \in \mathbb{R}, \tag{3.4}$$

$$f(t,x) \in [f_1(t,x,u(t,x))); f_2(t,x,u(t,x))], \quad x \in \Omega, t \in \mathbb{R}, \tag{3.5}$$

$$u(t+T) = -u(t), \qquad x \in \Omega, t \in \mathbb{R}, \tag{3.6}$$

$$u(t,x) = 0, \qquad t \in \mathbb{R},\ x \in \partial\Omega, \tag{3.7}$$

where $\lambda > 0$. Let

$$X = C(\overline{\Omega}), \quad X_0 = C_0(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\}.$$

Let $f_i : \mathbb{R} \times C_0(\overline{\Omega}) \to C(\overline{\Omega})$, for $i = 1, 2$, as follows

$$f_i(t, v)(x) = \tilde{f}_i(t, x, v(x)),$$

where $\tilde{f}_i : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies:

(H1) $\tilde{f}_i(\cdot, x, z)$ is measurable for every $x \in \Omega$; $\tilde{f}_i(t, \cdot, z)$ is continuous for each $t, z \in \mathbb{R}$, and $\tilde{f}_i(t, x, \cdot)$ is continuous for all $t \in \mathbb{R}$ and $x \in \Omega$;

(H2) $|\tilde{f}_i(t, x, z)| \leq m(t)(|z| + 1)$, for all $t, z \in \mathbb{R}, x \in \Omega$, where $m \in L^1_{loc}(\mathbb{R}; \mathbb{R}^+)$;

(H3) $\tilde{f}_i(t + T, x, -z) = -\tilde{f}_i(t, x, z)$, for all $t, z \in \mathbb{R}, x \in \Omega$.

Following Theorem 3.1, problem (3.4)-(3.7) have $T-$anti-periodic solutions provided that

$$\frac{2}{1 - e^{-\lambda T}} \int_0^T m(s)ds < 1.$$

3.3.2. Example 2

We consider the following problem

$$\partial_t u(t, x) = \sum_{k,l=1}^{M} \partial_k(a_{kl}(x)\partial_l)u(t, x) + a_0(x)u(t, x) + f(t, x, u(t, x)),$$
$$x \in \Omega, t \in \mathbb{R}, \tag{3.8}$$

$$f(t, x) \in [f_1(t, x, u(t, x))); f_2(t, x, u(t, x))], \quad x \in \Omega, t \in \mathbb{R}, \tag{3.9}$$

$$u(t + T) = -u(t), \qquad x \in \Omega, t \in \mathbb{R}. \tag{3.10}$$

$$\sum_{k,l=1}^{M} n_k(x)a_{kl}(x)\partial_l u(t, x) = 0, \ t \in \mathbb{R}, x \in \partial\Omega. \tag{3.11}$$

Here $\Omega \subseteq \mathbb{R}^M$ is a bounded domain with boundary $\partial\Omega$ of class C^2 and $n(x)$ is the outer unit normal vector. We assume that:

$$a_{kl} \in C^1(\overline{\Omega}), \quad k, l = 1, \cdots, M, \quad a_0 \in C(\overline{\Omega}),$$

where $\sum\limits_{k,l=1}^{n} a_{kl}(x) v_k v_l \geq \eta |v|^2$, for a constant $\eta > 0, x \in \overline{\Omega}, v \in \mathbb{R}^n$.
On $X = L^p(\Omega), 1 < p < \infty$, we consider the operator

$$A(x, D) := \sum_{k,l=1}^{M} \partial_k (a_{kl}(x) \partial_l) + a_0(x)$$

with domain

$$D(A) := \big\{ f \in \bigcap_{p>1} W^{2,p}(\Omega) : \\ A(\cdot, D) f \in C(\overline{\Omega}), \sum_{k,l=1}^{M} n_k(\cdot) a_{kl}(\cdot) \partial_l f = 0 \text{ on } \partial\Omega \big\}.$$

By Schnaubelt (2001), A generates a hyperbolic semigroup $T(\cdot)$ on X with the constants $M, \lambda > 0$.
Let $f_i : \mathbb{R} \times L^p(\Omega) \to L^p(\Omega)$, for $i = 1, 2$, as follows

$$f_i(t, v)(x) = \tilde{f}_i(t, x, v(x)),$$

where $\tilde{f}_i : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies:

(H4) $\tilde{f}_i(\cdot, \cdot, z)$ is measurable for each $t, z \in \mathbb{R}$; $\tilde{f}_i(t, x, \cdot)$ is continuous for a.e. $t \in \mathbb{R}$ and $x \in \Omega$;

(H5) $|\tilde{f}_i(t, x, z)| \leq \tilde{m}(t)(|z| + 1)$, for all $t, z \in \mathbb{R}, x \in \Omega$, where $\tilde{m} \in L^1_{loc}(\mathbb{R}; \mathbb{R}^+)$;

(H6) $|\tilde{f}_i(t, x, z) - \tilde{f}_i(t, x, z')| \leq k(t)|z - z'|$, where $k \in L^1_{loc}(\mathbb{R}; \mathbb{R}^+)$,

(H7) $\tilde{f}_i(t + T, x, -z) = -\tilde{f}_i(t, x, z)$, for all $t, z \in \mathbb{R}, x \in \Omega$.

We have the following result due to Theorem 3.1.

Theorem 3.2. *Problem* (3.8)-(3.11) *have* $T-$*anti-periodic solution provided that*

$$\frac{2M}{1-e^{-\lambda T}}\int_0^T \tilde{m}(s)ds < 1.$$

Chapter 4

WEAK STABILITY FOR A CLASS OF FRACTIONAL DIFFERENTIAL INCLUSIONS

We propose a unified approach to prove the global solvability and weakly asymptotic stability for a semilinear fractional differential inclusion subject to impulsive effects and nonlocal condition.
The content of this chapter is written based on the papers [1] and [3] in the author's works related to the thesis that has been published.

4.1. Setting problem

Let $(X, \|\cdot\|)$ be a Banach space. Consider the following problem

$$^{C}D_0^{\alpha}u(t) \in Au(t) + F(t, u(t), u_t),\ t > 0, t \neq t_k, k \in \Lambda, \tag{4.1}$$

$$\Delta u(t_k) = I_k(u(t_k)), \tag{4.2}$$

$$u(s) + g(u)(s) = \varphi(s),\ s \in [-h, 0], \tag{4.3}$$

where $D_0^{\alpha}, \alpha \in (0,1)$, is the fractional derivative in the Caputo sense, A is a closed linear operator in X which generates a strongly continuous semigroup $W(\cdot)$, F is a multivalued map, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $k \in \Lambda \subset \mathbb{N}$, $\inf_{k\in\Lambda}(t_{k+1} - t_k) > 0$, I_k and g are the continuous functions. Here u_t stands for the history of the state function up to the time t.
Let $\Sigma(\varphi)$ be the solution set of (4.1)-(4.3) with respect to the initial datum φ such that $0 \in \Sigma(0)$. The zero solution of (4.1)-(4.3) is said to be weakly asymptotically stable if it is

1) *stable*: for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\|\varphi\|_h < \delta$ then $\|u_t\|_h < \epsilon$ for any $u \in \Sigma(\varphi)$, here $\|\cdot\|_h$ denotes the norm in $C([-h, 0]; X)$;

2) *weakly attractive*: for any $\varphi \in \mathcal{B}$, there exists $u \in \Sigma(\varphi)$ satisfying $\|u_t\|_h \to 0$ as $t \to +\infty$.

4.2. Functional space and measure of noncompactness

We denote by $\mathcal{E} = PC(J; X)$ the space of piecewise continuous functions defined on $J \subset \mathbb{R}$ and take values in X. If $J = [-h, +\infty)$, we consider the following space

$$PC_\varrho([-h, +\infty); X) = \{u \in PC([-h, +\infty); X) : \lim_{t \to +\infty} \frac{u(t)}{\varrho(t)} = 0\},$$

where $\varrho : \mathbb{R}^+ \to [1, +\infty)$ is a continuous and nondecreasing function. We have $PC_\varrho([-h, +\infty); X)$ with the norm

$$\|u\|_\varrho = \sup_{t \in [-h,0]} \|u(t)\| + \sup_{t \geq 0} \frac{\|u(t)\|}{\varrho(t)},$$

is a Banach space. We will define a new regular MNC in this space as follows

$$\chi^*(D) = \sup_{T>0} \chi_{PC}(\pi_T(D)) + \lim_{T \to +\infty} \sup_{u \in D} \sup_{t \geq T} \frac{\|u(t)\|}{\varrho(t)}, \tag{4.4}$$

where $\pi_T(u)$ is the restriction of u to $[-h, T]$, and $D \subset PC_\varrho$.

4.3. Existence of solutions on the half line

We assume that:

(**A**) *The C_0-semigroup $\{W(t)\}_{t \geq 0}$ generated by A is norm-continuous and $\|W(t)x\| \leq M_A \|x\|, \forall t \geq 0, x \in X$.*

(**F**) *The nonlinearity $F : \mathbb{R}^+ \times X \times C([-h, 0]; X) \to X$ satisfies:*

1) *$F(\cdot, v, w)$ is u.s.c for each $t \in \mathbb{R}^+$;*
2) *the multi-valued map $t \mapsto F(t, u(t), u_t)$ admits a strongly measurable selection for each $u \in PC_\varrho$;*
3) *there exist functions $m \in L^p_{loc}(\mathbb{R}^+)$ such that*

$$\|F(t, v, w)\| = \sup\{\|\xi\| : \xi \in F(t, v, w)\} \leq m(t)(\|v\| + \|w\|_h),$$

for all $(t, v, w) \in \mathbb{R}^+ \times X \times C([-h, 0]; X)$, here $\|\cdot\|_h$ is the norm in $C([-h, 0]; X)$;

4) if $W(\cdot)$ is noncompact, there exists a function $k \in L^p_{loc}(\mathbb{R}^+)$ such that

$$\chi(F(t,V,W)) \leq k(t)\left[\chi(V) + \sup_{t\in[-h,0]} \chi(W(t))\right],$$

for a.e. $t \in \mathbb{R}^+$, and $\forall V \in \mathcal{B}(X), W \in \mathcal{B}(C([-h,0];X))$.

*(**G**) The function $g : PC_\varrho \to C([-h,0];X)$ is continuous and satisfies*

1) $\|g(u)\|_h \leq \Psi_g(\|u\|_\varrho)$ for all $u \in PC_\varrho$, where Ψ_g is a function on $\mathbb{R}^+$;

2) $\exists \eta \geq 0$ such that $\chi_h(g(D)) \leq \eta \cdot \chi_\infty(D)$ for all $D \in \mathcal{B}(PC_\varrho)$, where χ_h is the Hausdorff MNC in $C([-h,0];X)$.

*(**I**) The function $I_k : X \to X, k \in \Lambda$, is continuous and satisfies:*

1) there exists a nonnegative sequence $\{l_k\}_{k\in\Lambda}$ such that $\sum_{k\in\Lambda} l_k < \infty$ and

$$\|I_k(x)\| \leq l_k \|x\|, \text{ for all } x \in X, k \in \Lambda,$$

2) there exists a nonnegative sequence $\{\mu_k\}_{k\in\Lambda}$ such that

$$\chi(I_k(B)) \leq \mu_k \chi(B), \forall B \in \mathcal{B}(X).$$

For $u \in PC_\varrho$ we denote

$$\mathcal{P}^p_F(u) = \{f \in L^p_{loc}(\mathbb{R}^+;X) : f(t) \in F(t,u(t),u_t) \text{ for a.e. } t \in \mathbb{R}^+\}.$$

Definition 4.1. A function $u : [-h,+\infty) \to X$ is said to be an integral solution of problem (4.1)-(4.3) if and only if $u(t) + g(u)(t) = \varphi(t)$ for $t \in [-h,0]$, and $\exists f \in \mathcal{P}^p_F(u)$ such that for any $t > 0$

$$u(t) = S_\alpha(t)[\varphi(0) - g(u)(0)] + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(u(t_k)) + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)ds.$$

We have following theorem.

Theorem 4.1. *For $\sigma \in (0,1)$, assume that*

$$(1+M_A)\liminf_{r\to+\infty}\frac{\Psi_g(r)}{r}+M_A\sum_{k\in\Lambda}l_k$$
$$+2\sup_{t>0}\int_0^t\frac{(t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)}{\varrho(t-s)}ds<1,$$

$$\ell=\eta M_A+M_A\sum_{k\in\Lambda}\mu_k+8\sup_{t\geq 0}\int_0^t(t-s)^{\alpha-1}\|P_\alpha(t-s)\|_\chi k(s)ds<1,$$
$$\vartheta=\sup_{t>0}\int_0^{\sigma t}\frac{\|P_\alpha(t-s)\|}{\varrho(t-s)}m(s)ds<\infty,$$
$$\kappa=\sup_{t>0}\int_{\sigma t}^t\frac{(t-s)^{\alpha-1}\|P_\alpha(t-s)\|}{\varrho(t-s)}m(s)ds<\frac{1}{2}.$$

Then problem (4.1)-(4.3) *has at least one integral solution in PC_ϱ.*

4.4. Weak stability result

We replace (**A**), (**F**) and (**G**) by stronger ones:

(**A** *) *The semigroup $W(\cdot)$ is norm-continuous and there exists $\beta>0$ such that*

$$\|W(t)x\|\leq M_Ae^{-\beta t}\|x\|,\forall t\geq 0,x\in X.$$

(**F** *) *The function F satisfies (**F**) with $m\in L^1(\mathbb{R}^+)\cap L^p_{loc}(\mathbb{R}^+)$.*

(**G** *) *The nonlocal function g satisfies (**G**) with $\Psi_g(r)=\nu\cdot r,\forall r\geq 0$, here ν is a positive constant.*

Theorem 4.2. *Let (**A** *), (**F***), (**G***) and (**I**) hold. Then the zero*

solution of (4.1)-(4.3) *is weakly asymptotically stable provided that*

$$\ell = \eta M_A + M_A \sum_{k\in\Lambda} \mu_k + 8 \sup_{t\geq 0} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\|_\chi k(s) ds < 1,$$

$$\varpi = (1+M_A)\nu + M_A \sum_{k\in\Lambda} l_k + 2 \sup_{t>0} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s) ds < 1.$$

4.5. Application

We consider the following lattice differential system

$$\frac{d^\alpha}{dt^\alpha} u_i(t) = (Au(t))_i + f_i(t),\ t > 0, t \neq t_k, k \in \mathbb{N}, \tag{4.5}$$

$$f_i(t) \in [f_{1i}(t, u_i(t), u_i(t-\rho(t))), f_{2i}(t, u_i(t), u_i(t-\rho(t)))], \tag{4.6}$$

$$\Delta u_i(t_k) = I_{ik}(u_i(t_k)), \tag{4.7}$$

$$u_i(s) + \sum_{j=1}^{N} c_j u_i(\tau_j + s) = \varphi_i(s), s \in [-h, 0], \tau_j > 0, \tag{4.8}$$

where $u = (u_i) : [-h, +\infty) \to \ell^2$ is the unknown function, $\dfrac{d^\alpha}{dt^\alpha}$ is the Caputo derivative of order $\alpha \in (0,1)$, $A : \ell^2 \to \ell^2$ is defined as follows $(Av)_i = v_{i+1} - (2+\lambda)v_i + v_{i-1},\ v \in \ell^2$, $\rho : \mathbb{R}^+ \to [0, h]$ is a continuous function, $\lambda > 0$. We give the following assumptions

(N1) $f_{1i}, f_{2i} : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}, i \in \mathbb{Z}$, are continuous and satisfy

$$\max\{|f_{1i}(t,y,z)|^2, |f_{2i}(t,y,z)|^2\} \leq m^2(t)(|y|^2 + |z|^2),$$

for all $(t, \eta, z) \in \mathbb{R}^+ \times \mathbb{R}^2$, where $m \in C(\mathbb{R}^+; \mathbb{R}^+)$ satisfies

$$m(t) \leq \frac{C_m}{1+t^{\alpha+1}} \text{ for some } C_m > 0.$$

(N2) $I_{ik} : \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{Z}, k \in \mathbb{N}$, are continuous such that

$$|I_{ik}(y)| \leq l_k |y|,$$

where $l_k > 0, \forall k \in \mathbb{N}$ such that $\sum_{k\in\mathbb{N}} l_k < \infty$.

If the hypotheses (N1) and (N2) hold, we get the weakly asymptotic stability of zero solution to (4.5)-(4.8).

4.6. Special case

We consider a special case of the problem (4.1)-(4.3), when F is a singleton function, denoted by f.

$$^{C}D_0^{\alpha}u(t) = Au(t) + f(t, u(t), u_s),\ t \neq t_k, t_k > 0, k \in \Lambda, \tag{4.9}$$

$$\Delta u(t_k) = I_k(u(t_k)), \tag{4.10}$$

$$u(s) + g(u)(s) = \varphi(s),\ s \in [-h, 0]. \tag{4.11}$$

We assume that:

(Aa) *$W(\cdot)$ is norm continuous and $\exists \beta > 0$ such that*

$$\|W(t)x\| \leq M_A e^{-\beta t}\|x\|, \forall t \geq 0, x \in X.$$

(Fa) *$f(\cdot, v, w)$ is measurable for each $v \in X$, $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in \mathbb{R}^+$, $f(t, 0, 0) = 0$, and $\exists k \in L^p(\mathbb{R}^+), p > \frac{1}{\alpha}$, such that: for all $v_1, v_2 \in X, w_1, w_2 \in \mathcal{C}_h$*

$$||f(t, v_1, w_1) - f(t, v_2, w_2)|| \leq k(t)(||v_1 - v_2|| - ||w_1 - w_2||_{\mathcal{C}_h}), t \in \mathbb{R}^+.$$

(Ga) *g is continuous, $g(0) = 0$ and $\exists \eta > 0$ such that*

$$||g(w_1) - g(w_2)||_h \leq \eta ||w_1 - w_2||_\infty, \forall w_1, w_2 \in \mathcal{PC}_0.$$

(Ia) *$I_k, k \in \Lambda$, is continuous, $I_k(0) = 0$ and $\exists \{\mu_k\}_{k \in \Lambda}$ such that*

$$||I_k(x) - I_k(y)|| \leq \mu_k ||x - y||, \text{ for all } x, y \in X.$$

Since Banach contraction principle, we have following theorem.

Theorem 4.3. *Let* (**Aa**), (**Fa**), (**Ga**), (**Ia**) *hold. Then problem* (4.9)-(4.11) *has a unique solution $\|u(t)\| = o(1)$, provided that*

$$\Big(\eta + \sum_{k \in \Lambda} \mu_k\Big) M_A + 2 \sup_{t \geq 0} \int_0^t (t - s)^{\alpha - 1} \|P_\alpha(t - s)\| k(s) ds < 1.$$

CONCLUSION

1. RESULTS

This thesis has studied the asymptotic behavior of solutions to some classes of evolution inclusions in Banach space. The results are:

1) For the class of differential inclusions with finite delay whose linear part generates an integrated semigroup: We have proved the global solvability and the existence of a compact global attractor for the $m-$semiflow generated by our system.

2) For a class of polytope differential inclusions with Hille–Yosida operator: We have proved the existence of anti-periodic solutions.

3) For a semilinear fractional differential inclusion subject to impulsive effects and nonlocal condition: We have proved the global solvability and weakly asymptotic stability of zero solution. In a special case, the nonlinearity F is singleton and satisfies the Lipschitz condition, we have proved the existence and uniqueness of decay solution.

2. RECOMMENDATION

Some open problems are

- Study the asymptotic behavior of solutions (using theory of global/pullback attractor or stability theory) to some classes of differential inclusions with delay variation or infinite delay.

- Study the existence of some classes of special solutions such as periodic, anti-periodic and decay solutions to some other classes of evolution inclusions.

AUTHOR'S WORKS RELATED TO THE THESIS THAT HAVE BEEN PUBLISHED

1) T.D. Ke, D. Lan, Decay integral solutions for a class of Impulsive Fractional differential equations in Banach spaces, *Fractional Calculus and Applied Analysis*, Volume 17, Number 1 (2014) 96-121.

2) T.D. Ke, D. Lan, Global attractor for a class of functional differential inclusions with Hille–Yosida operators, *Nonlinear Analysis: Theory, Methods and Applications*, Volume 103 (2014) 72–86.

3) T.D. Ke, D. Lan, Generalized Cauchy problem governed by fractional differential inclusions on the half line, *submitted* .

4) T.D. Ke, D. Lan, Existence of Anti-periodic solutions for a class of polytope differential inclusions with Hille-Yosida operators, *submitted* .

Results of the thesis have been reported at:

1) 8th National Mathematics Congress, Nha Trang, 08/2013;

2) Seminar of Department of Analysis, Falcuty of Mathematics, Hanoi National University of Eduacation;

3) Seminar of Department of Differential Equations, Institute of Mathematics, VAST.

4) Seminar of Department of Optimization and Control Theory, Institute of Mathematics, VAST.

5) Seminar on Asymptotic Behavior of Solutions to Differential Equations and Applications, School of Applied Mathematics and Informatics, Hanoi University of Science and Technology.